MODELLING OF 2-D AR FIELDS WITH THE QUARTER-PLANE LATTICE FILTERS

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A new lattice filter structure which is the generalization of the filter developed by Parker and Kayran is proposed to model two-dimensional (2-D) autoregressive (AR) fields. Four prediction error fields whose linear combination defines the first stage are generated from the input data field. After the first stage, two additional error fields are generated using one of the backward prediction error fields and the higher-order stages are defined by the linear combination of these six error fields, and the reflection coefficients are calculated. This improved the performance which is confirmed by computer simulations. In addition, a recursive relationship between the reflection coefficients and the AR coefficients is derived.

INTRODUCTION

4.1 Background Material

Theory of one-dimensional (1-D) lattice filters is well-known and well-developed. However, there have been very few approaches to the 2-D lattice filter and its use in 2-D spectral estimation [1-5].

A basic approach to the modelling of 2-D fields by the reflection coefficients was made by Marzetta [1].

Parker and Kayran [2] have introduced the concept of four prediction error fields which are combined into a quarter-plane 2-D lattice filter structure, which is then used for spectral estimation. Semifront [3] has further extended this approach.

Parker et al. [3], have extended the quarter-plane model to asymmetric half-plane where five prediction error fields involving six reflections coefficients are defined.

1.2 2-D Lattice Filter

The quarter-plane 2-D lattice filters are directly developed from the 1-D causal lattice filters, where one forward and three backward prediction error fields are generated in a single structure [2]. The lattice filter consists of successive stages. The output relation of stage \( m = (m_1, m_2) \) of the lattice filter can be obtained from stage \( m = (m_1 - 1, m_2 - 1) \) as follows [2]:

\[
\begin{bmatrix}
    f(m_1, n_1, n_2) \\
    b_0(m_1, n_1, n_2) \\
    b_1(m_1, n_1, n_2) \\
    b_2(m_1, n_1, n_2)
\end{bmatrix} = \Gamma^{(m)} \begin{bmatrix}
    f(m_1 - 1, n_1, n_2) \\
    b_0(m_1 - 1, n_1, n_2) \\
    b_1(m_1 - 1, n_1, n_2) \\
    b_2(m_1 - 1, n_1, n_2)
\end{bmatrix}
\]

where \( \Gamma^{(m)} \) is the coefficient matrix of \( (m_1, m_2) \) stage and is given as follows:

\[
\Gamma^{(m)} = \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    \gamma_{10}^{(m_1, m_2)} & \gamma_{11}^{(m_1, m_2)} & \gamma_{12}^{(m_1, m_2)} & \gamma_{13}^{(m_1, m_2)} \\
    \gamma_{20}^{(m_1, m_2)} & \gamma_{21}^{(m_1, m_2)} & \gamma_{22}^{(m_1, m_2)} & \gamma_{23}^{(m_1, m_2)} \\
    \gamma_{30}^{(m_1, m_2)} & \gamma_{31}^{(m_1, m_2)} & \gamma_{32}^{(m_1, m_2)} & \gamma_{33}^{(m_1, m_2)}
\end{bmatrix}
\]

This lattice filter, as seen from Eq. (2), has three parameters at each stage and it will be referred to as the three-parameter lattice filter. Thus starting with zero-order lattice \( m = 0 \), four prediction error fields are generated from the input field; namely, \( f(0, n_1, n_2) = b_0(0, n_1, n_2) = b_1(0, n_1, n_2) = b_2(0, n_1, n_2) = u(n_1, n_2) \). Their linear combination is used to calculate the prediction error fields of the higher order stages.

Ideally, if the input signal to the lattice filter is a 2-D AR process, the coefficients of the process will be recovered exactly if there is one-to-one correspondence between the lattice parameters and data support. However, the number of samples in the data support is related to the \((M,M)\)th order lattice filter as \((M+1)^2-1\). The sufficient number of lattice parameters for exact modelling of the 2-D AR field is one less than the size of the data support and related to the order of the lattice filter as \((M+1)^2-1\). However, the number of actually calculated parameters grows linearly with the order, namely as 3M. These concepts are illustrated in Fig. 1. Thus, the three-parameter lattice filter is not sufficient to represent all classes of 2-D AR fields. It can represent only a subset of such fields. This is the result of quadratic growth of data support compared to the linear growth of reflection coefficients with the filter order. 2-D AR fields can be modelled more accurately introducing...
more parameters, thus a new structure is presented in the next section.

Fig. 1 Data support-lattice parameter relation

2. EXTENDED LATTICE FILTER

The extended lattice filter [4,5] is developed as an extension of the three-parameter lattice filter.

The first stage of the extended filter has three coefficients which is equal to the required number of coefficients. For the second stage, eight parameters are needed which implies that five new parameters should be calculated. Increasing the number of error fields by two, the required eight parameters can be obtained. This extension is introduced after the first stage and it is achieved by generating two additional backward prediction error fields, namely \( b_{21}(1;n_1,n_2) \) and \( b_{12}(1;n_1,n_2) \). These new error fields are generated from the backward prediction error field \( b_{11}(1;n_1,n_2) \) as follows:

\[
\begin{align*}
    b_{21}(1;n_1,n_2) &= b_{11}(1;n_1,n_2) \\
    b_{12}(1;n_1,n_2) &= b_{11}(1;n_1,n_2)
\end{align*}
\]

Thus at the input of the second stage and consequently of the higher order stages, there will be six error fields, where one of them is the forward and the rest are the backward prediction error fields. Their linear combination will define the next lattice stage.

The input/output relation of the first stage is still given by Eq. (1), however the input/output relations of the higher order stages are given as follows:

\[
\begin{align*}
    \begin{bmatrix}
    f(m;n_1,n_2) \\
    b_{10}(m;n_1,n_2) \\
    b_{11}(m;n_1,n_2) \\
    b_{01}(m;n_1,n_2) \\
    b_{21}(m;n_1,n_2) \\
    b_{12}(m;n_1,n_2)
    \end{bmatrix}
    &= \Gamma(m)
    \begin{bmatrix}
    f(m-1;n_1,n_2) \\
    b_{10}(m-1;n_1-1,n_2) \\
    b_{11}(m-1;n_1-1,n_2-1) \\
    b_{01}(m-1;n_1,n_2-1) \\
    b_{21}(m-1;n_1-1,n_2) \\
    b_{12}(m-1;n_1,n_2-1)
    \end{bmatrix}
\end{align*}
\]

The coefficient matrix of the extended lattice filter is given as follows where the indices of the entries are dropped for convenience:

\[
\Gamma(m) = \begin{bmatrix}
    1 & -\Gamma_{10} & -\Gamma_{11} & -\Gamma_{01} & -\delta_{21} & -\delta_{12} \\
    -\Gamma_{10} & 1 & -\Gamma_{11} & -\delta_{10} & -\delta_{22} & -\delta_{12} \\
    -\Gamma_{11} & -\Gamma_{01} & 1 & -\delta_{20} & -\delta_{22} & -\delta_{12} \\
    -\Gamma_{01} & -\Gamma_{11} & -\delta_{20} & 1 & -\delta_{22} & -\delta_{12} \\
    -\delta_{21} & -\delta_{10} & -\delta_{20} & -\delta_{22} & 1 & -\delta_{22} \\
    -\delta_{12} & -\delta_{02} & -\delta_{11} & -\delta_{01} & -\delta_{22} & 1
\end{bmatrix}
\]

In general, the input/output relation of the extended lattice filter is given as follows:

\[
\varepsilon(m;n_1,n_2) = \Gamma(m) \tilde{\varepsilon}(m-1;n_1,n_2)
\]

where \( \varepsilon(m;n_1,n_2) \) and \( \tilde{\varepsilon}(m;n_1,n_2) \) are prediction error field vectors of stages \( (m,n_2) \) and \( (m-1,n_2-1) \), respectively. Note the shifts in the elements of vector \( \tilde{\varepsilon}(m-1;n_1,n_2) \).

Table 1 shows the dimensions of the error vectors and the coefficient matrices for different stages of the extended lattice filter.

<table>
<thead>
<tr>
<th>TABLE 1</th>
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<tbody>
<tr>
<td>Dimensions of vectors and matrices</td>
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<tr>
<td>------------------------------------------</td>
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<tr>
<td>Lattice Stages</td>
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<td>( \tilde{\varepsilon}(m-1;n_1,n_2) )</td>
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<tr>
<td>( \varepsilon(m;n_1,n_2) )</td>
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<tr>
<td>( \Gamma(m) )</td>
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<td>( \chi(m) )</td>
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In order to calculate the lattice parameters at each stage, the prediction error fields should be minimized in the mean-squared sense. The optimization can be performed in one of the fields (that is either in the forward prediction error field or in one of the backward prediction error fields) or in all the error fields at the same time. To do this optimization, the following expression for the mean-squared prediction error has to be solved:

\[
P(m) = \mathbb{E}[\varepsilon^T(m;n_1,n_2) \varrho \varepsilon(m;n_1,n_2)]
\]

\[
= \mathbb{E}[\tilde{\varepsilon}^T(m-1;n_1,n_2) \gamma(m) \varrho \tilde{\varepsilon}(m-1;n_1,n_2)]
\]

where \( \mathbb{E}[\cdot] \) is the expectation operator and \( \varrho \) is a diagonal matrix. The diagonal elements \( (\varrho_s)'s \) can take on values of either 1 or 0, where a 1 on the \( i^{th} \) diagonal element implies the optimization is done in the \( i^{th} \) field.

Minimum mean-squared solution of Eq. (7) will lead to the following expression for the optimum lattice filter parameters:

\[
R(m-1) \chi(m) = \Gamma(m-1)
\]
\[ \gamma(m) = [\Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_{m-1}, \Gamma_m] \]  

is the vector of lattice parameters of stage \((m, m)\). Note that for the first stage, \(\gamma\) consists of only the first three elements. \(R\) and \(g\) consist of the prediction error field correlations. Dimensions of \(R\), \(\gamma\) and \(e\) are also given in Table 1.

From Eq. (8), the optimum lattice parameters can easily be calculated by the inversion of matrix \(R\).

\[ \gamma(m) = R^{-1}(m-1) R(m-1) \]  

Using the foregoing results, a simple recursive equation for the mean-squared prediction error from stage to stage is derived. Simplifying Eq. (7) yields:

\[ P(m) = P(m-1) - \gamma^2(m) R(m-1) \gamma(m) \]  

If the correlation matrix \(R\) is positive definite, minimum mean-squared error will decrease as the number of lattice stages increase. However, stability is not guaranteed in this procedure.

There is a recursive relationship between the lattice parameters and the AR coefficients of the input data field. This recursive relation is given as follows for \(m \geq 2\):

\[ \begin{bmatrix} \theta_0(m) \\ \theta_1(m) \\ \theta_2(m) \end{bmatrix} = R(m) \begin{bmatrix} \theta_0(m) \\ \theta_1(m) \\ \theta_2(m) \end{bmatrix} \]  

Note that for \(m=1\), since \(R\) is given by Eq. (2) only the first four partitions of Eq. (12) should be used.

The augmented matrices are defined as follows:

\[ \begin{bmatrix} \tilde{A}_0(m) \\ \tilde{A}_1(m) \\ \tilde{A}_2(m) \end{bmatrix} = \begin{bmatrix} \theta_0(m) \\ \theta_1(m) \\ \theta_2(m) \end{bmatrix} \]  

As a consequence of Eqs. (3a) and (3b):

\[ \begin{aligned} \tilde{A}_0(1) &= \tilde{A}_1(1) \\ \tilde{A}_2(1) &= \tilde{A}_1(1) \end{aligned} \]  

After the lattice parameters are calculated, the AR coefficients can easily be found using Eqs. (12) and (13).

The extended lattice filter brings improvement in modeling second order AR fields. Due to modularity of the extended lattice filter, two additional backward prediction error fields can be generated at the output of each stage. As a result of this, sufficient number of parameters can be obtained and all order AR fields can then be modeled perfectly.

3. COMPUTER SIMULATIONS AND RESULTS

Computer simulations are performed to check the validity of the theory presented. In the simulations, the input data is a 2-D AR field given as follows:

\[ u(n_1, n_2) = \sum_{k_1, k_2} a(k_1, k_2) u(n_1-k_1, n_2-k_2) + \sigma^2 w(n_1, n_2) \]  

where \(w(n_1, n_2)\) is a white gaussian noise of variance \(\sigma^2\). The coefficients of the AR data field are modeled by the lattice reflections coefficients. The forward prediction error transfer function can be defined as follows:

\[ H(m; z_1, z_2) = \frac{E_{00}(m; z_1, z_2)}{E_{00}(0; z_1, z_2)} \]  

\[ = [1, z_1^{-1}, \ldots, z_2^{-m}]^T [\tilde{A}] [1, z_2^{-1}, \ldots, z_2^{-m}] \]

where the matrix \(\tilde{A}\) consists of either the original or the modeled AR field coefficients, namely \(a(k_1, k_2)\)'s. The transfer function of the modeled AR data field can be obtained by starting with the first order lattice filter and increasing the order until the desired order \(M\) is reached. Matrix \(\tilde{A}\) can be calculated from the reflection coefficients as follows:
1. Define $A_0$, $A_1$, $A_1^*$, $A_2$, $A_2^*$ as the forward and backward prediction error filter coefficient matrices.
2. Set $A_0(0) = A_1(0) = A_2(0) = A_2^*(0) = 1$
3. Set $m = (m_1, m_2)$
4. Find the reflection coefficients for stage one using Eq. (10).
5. Find the coefficient matrix of the first order AR field (for $m = 1$) using Eqs. (12) and (13a)-(13d).
6. Increase $m$ by one and compute the augmented matrices using Eqs. (13a)-(13f).
7. Calculate the reflection coefficients of stage $m$ using Eq. (10).
8. Compute the AR coefficient matrix of stage $m$ using Eqs. (12) and (13a)-(13f).
9. Repeat steps 6, 7, and 8 until the desired order $m$ is obtained.

**Example**

The coefficient matrix of the original AR data field is given as follows:

$$
A_{00} = \begin{pmatrix}
1.0000 & -0.1000 & 0.1000 & -0.1000 \\
-0.1000 & 0.1000 & -0.1000 & 0.1000 \\
0.1000 & -0.1000 & 0.1000 & -0.1000 \\
-0.1000 & 0.1000 & -0.1000 & 0.1000
\end{pmatrix}
$$

The coefficient matrix obtained at the output of the third order extended filter is given as follows:

$$
A_{00}(3) = \begin{pmatrix}
1.0000 & -0.0559 & 0.0632 & -0.1031 \\
-0.0637 & 0.0985 & -0.0791 & 0.1072 \\
-0.0594 & -0.1084 & 0.0704 & -0.0100 \\
-0.0918 & 0.1045 & -0.0085 & 0.0746
\end{pmatrix}
$$

In the example, the optimization is done in the forward field. The 3-D magnitude plots of the original data and the modelled data are given in Figs. 2 and 3.

**Fig. 2**  Magnitude plot of the original spectrum

**Fig. 3**  Magnitude plot of the spectrum approximated by the extended lattice filter

4. **Conclusions**

As seen from Figs. 2 and 3, the extended lattice filter models 2-D AR data field with high accuracy and it is able to distinguish the two spectral peaks clearly. The modelled spectrum exhibits all the characteristics of the input data very accurately. A good duplicate of the original spectrum is obtained by processing the data at the input.

**REFERENCES**


